

The Fixed Point of a Generalization of the Renormalization Group Maps for Self-Avoiding Paths on Gaskets

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Let $W(x, y) = ax^3 + bx^4 + f_5x^5 + f_6x^6 + (3ax^2)^2y + g_5x^5y + h_3x^3y^2 + h_4x^4y^2 + n_3x^3y^3 + a_{24}x^2y^4 + a_{05}y^5 + a_{15}xy^5 + a_{06}y^6$, and $X = \frac{\partial W}{\partial x}$, $Y = \frac{\partial W}{\partial y}$, where the coefficients are non-negative constants, with $a > 0$, such that $X^2(x, x^2) - Y(x, x^2)$ is a polynomial of x with non-negative coefficients.

Examples of the 2 dimensional map $\Phi : (x, y) \mapsto (X(x, y), Y(x, y))$ satisfying the conditions are the renormalization group (RG) maps (modulo change of variables) for the restricted self-avoiding paths on the 3 and 4 dimensional pre-gaskets.

We prove that there exists a unique fixed point (x_f, y_f) of Φ in the invariant set $\{(x, y) \in \mathbb{R}_+^2 \mid x^2 \geq y\} \setminus \{0\}$.

KEY WORDS: renormalization group, fixed point uniqueness, self-avoiding paths, Sierpinski gasket

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we study existence and uniqueness of fixed point for a 2 dimensional discrete time dynamical system in the first quadrant \mathbb{R}_+^2 , generated by the gradient

$$\Phi = (X, Y) = \text{grad } W = \left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y} \right) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2 \quad (1)$$

of a polynomial $W : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ with non-negative coefficients, such that the set $\Xi = \{(x, y) \in \mathbb{R}_+^2 \mid y \leq x^2\}$

is an invariant set of $\Phi = (X, Y)$.

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Let us state our main results. The first result deals with the existence of fixed point in the interior of Ξ .

Theorem 1. *Assume that $W : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the following:*

- (i) W is a polynomial in 2 variables x and y , each term of which has positive coefficient and of total degree 3 or more. Moreover, the term x^3 exists (i.e., the coefficient of x^3 is non-zero).
- (ii) Ξ of (2) is an invariant set of $\Phi = \text{grad } W$: if $(x, y) \in \Xi$, then $\Phi(x, y) \in \Xi$. Moreover, $Y(x, x^2) < X(x, x^2)^2$, $x > 0$, for $\Phi = (X, Y)$.
- (iii) There exists a term of the form $x^n y$ in W , i.e., the coefficient of $x^n y$ is non-zero for some $n \geq 2$.
- (iv) $R(x, z) = X(x, x^2 z)^2 - Y(x, x^2 z)$ is a polynomial in z , $1 - z$, and x , with non-negative coefficients. Namely, there exists a polynomial $\tilde{R}(x, z, s)$ in 3 variables with non-negative coefficients such that $R(x, z) = \tilde{R}(x, z, 1 - z)$. Moreover,

$$\frac{R(x, z)}{Y(x, x^2 z)} = O(x), \quad x \rightarrow 0, \quad (3)$$

where $O(x)$ is uniform in $z \in [0, 1]$.

Then there exists a fixed point (x_f, y_f) of Φ in the interior $\Xi^\circ = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0, x^2 > y\}$ of Ξ .

We note that Theorem 1 is not a direct consequence of standard topological fixed point theorems on Ξ , which allows for a fixed point on the boundary of Ξ , $\partial \Xi = \{(x, 0) \mid x \geq 0\} \cup \{(x, x^2) \mid x \geq 0\}$, which is trivial, because $(0, 0)$ is a fixed point of Φ under the conditions in the Theorem. We are looking for a fixed point in Ξ° , the interior of Ξ , not on the boundary.

We also note that restricting our attention to the subset $\Xi \subset \mathbb{R}_+^2$ is essential, because outside Ξ , fixed points may disappear and appear with small changes in the coefficients of W . For example, let $W_\epsilon(x, y) = \frac{1}{3}x^3 + x^4 y + \epsilon y^6$. (This choice satisfies the conditions in Theorem 1 and Theorem 2 below for $0 \leq \epsilon \leq 8/3$.) Then for positive ϵ , there are 4 fixed points of $\Phi_\epsilon = \text{grad } W_\epsilon$ in \mathbb{R}_+^2 : $(0, 0)$, $(x_1, y_1) = (0.662 \cdots + O(\epsilon), 0.192 \cdots + O(\epsilon))$, $(0, (6\epsilon)^{-1/4})$, and one of order $(O(\epsilon^{1/8}), O(\epsilon^{-1/4}))$, while for $\epsilon = 0$ the last 2 are absent and we have only 2 fixed points.

An intuition for the specific conditions on W in Theorem 1 arises in an attempt to extend a corresponding simple fact for function with 1 variable. Let $f(x)$ be a polynomial with non-negative coefficients with lowest order term of x^3 . Then there is a unique fixed point x_f of f' on the positive x axis ($f'(x_f) = x_f > 0$). Note that this is not the direct consequence of a standard topological fixed point theorem on an obvious invariant set $\mathbb{R}_+ = \{x \geq 0\}$ of the map f' , because $x = 0$

is a fixed point. Rather, the unique existence of the fixed point $x_f > 0$ is due to the positivity of the coefficients in the map. (Note also that if f further contains a term x^2 then $f'(x) \geq 2x > x$, and there are no positive fixed points, hence the orders in x of the terms are important for existence.) A simple way of extending this fact to 2 variables would be to assume that the second variable y is of order x^2 , at least for small x , and that this relation is preserved under the map $\Phi = (X, Y)$ in consideration. This motivates the non-negativity of the coefficients and conditions on $R = X^2 - Y$ in Theorem 1. We have added a couple of conditions to exclude fixed points on the boundary on $\partial \Xi \setminus \{0\}$ to avoid complications.

We turn to our second result, which is on the uniqueness of the fixed point (x_f, y_f) of Φ in Ξ^o . This is a more difficult problem than the existence result, and we have results only with 12 adjustable coefficients for W , in contrast to Theorem 1 which allows for indefinitely many terms.

Theorem 2. *Let $W : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be a polynomial defined by*

$$\begin{aligned} W(x, y) = & a x^3 + b x^4 + f_5 x^5 + f_6 x^6 + (3 a x^2)^2 y + g_5 x^5 y \\ & + h_3 x^3 y^2 + h_4 x^4 y^2 \\ & + n_3 x^3 y^3 + a_{24} x^2 y^4 + a_{05} y^5 + a_{15} x y^5 + a_{06} y^6, \end{aligned} \tag{4}$$

where all the constants $a, b, f_5, f_6, g_5, h_3, h_4, n_3, a_{24}, a_{05}, a_{15}, a_{06}$, are non-negative, and $a > 0$, and $R(x, z) = X(x, x^2 z)^2 - Y(x, x^2 z)$ is a polynomial in $z, 1 - z$, and x , with non-negative coefficients, in the same sense as in the corresponding condition in Theorem 1. Then there exists a unique fixed point (x_f, y_f) of $\Phi = \text{grad } W$ in Ξ^o .

The condition on R in Theorem 2 can be made explicit.

Proposition 3. *The conditions on W in Theorem 2 is equivalent to the following: W is as in (4), with the coefficients being non-negative, $a > 0$, and $R_n \geq 0, 5 \leq n \leq 10$, where the R_n s are defined by*

$$\begin{aligned} R_5 &= 24 a b - g_5 - 2 h_3, \\ R_6 &= 16 b^2 + 30 a f_5 - 2 h_4, \\ R_7 &= 216 a^3 + 40 b f_5 + 36 a f_6 - 3 n_3, \\ R_8 &= 288 a^2 b + 25 f_5^2 + 48 b f_6 + 30 a g_5 + 18 a h_3 - 5 a_{05} - 4 a_{24}, \\ R_9 &= 360 a^2 f_5 + 60 f_5 f_6 + 40 b g_5 + 24 b h_3 + 24 a h_4 - 5 a_{15}, \\ R_{10} &= 648 a^4 + 216 a^2 f_6 + 18 f_6^2 + 25 f_5 g_5 + 15 f_5 h_3 + 16 b h_4 \\ &\quad + 9 a n_3 - 3 a_{06}. \end{aligned}$$

That this is necessary is easily seen, if one explicitly writes the coefficients of x^n in $R(x, 1)$ for $5 \leq n \leq 10$. That the conditions in Proposition 3 are sufficient is proved by looking into the coefficients of x^n in $R(x, z)$ (each of which is a polynomial in z). It turns out that with W of the form (4), terms with x^n appear for $5 \leq n \leq 20$, among which no explicit negative signs appear for $n \geq 11$, hence the condition hold automatically, and for the remaining $5 \leq n \leq 10$, the power of z in the terms with negative signs are larger than any of the terms with positive signs, hence with the non-negativity conditions at $z = 1$, assumed in Proposition 3, it is straightforward to find a polynomial in z and $1 - z$ with non-negative coefficients. Proposition 3 is thus proved.

Among the examples of W satisfying the conditions in Theorem 2, or equivalently, in Proposition 3, are those related to the renormalization group (RG) maps for the restricted self-avoiding paths on the 3 and 4 dimensional pre-gaskets:^(5,6,8)

$$\begin{aligned}
 W_3(x, y) &= \frac{1}{3}x^3 + \frac{1}{2}x^4 + \frac{2}{5}x^5 + x^4y + 2x^3y^2 + \frac{22}{5}y^5, \\
 W_4(x, y) &= \frac{\sqrt{3}}{9}x^3 + \frac{1}{4}x^4 + \frac{2\sqrt{3}}{15}x^5 + \frac{1}{9}x^6 + \frac{1}{3}x^4y \\
 &\quad + \frac{2\sqrt{3}}{9}x^5y + \frac{2\sqrt{3}}{9}x^3y^2 + \frac{13}{18}x^4y^2 + \frac{32\sqrt{3}}{81}x^3y^3 + \frac{22}{27}x^2y^4 \quad (5) \\
 &\quad + \frac{22}{135}y^5 + \frac{44\sqrt{3}}{81}xy^5 + \frac{31}{81}y^6.
 \end{aligned}$$

It is straightforward to see that W_3 and W_4 satisfy all the conditions in Proposition 3. The fixed point equation $(x, y) = \Phi(x, y)$ for $\Phi = \text{grad } W_3$ is

$$\begin{aligned}
 x &= x^2 + 2x^3 + 2x^4 + 4x^3y + 6x^2y^2, \\
 y &= x^4 + 4x^3y + 22y^4,
 \end{aligned}$$

which coincides with that for $\vec{\Phi}$ in [Ref. 5, (2.3) and (2.4)], and the fixed point equation $(x, y) = \Phi(x, y)$ for $\Phi = \text{grad } W_4$ is, with the change of variables $x = \sqrt{3}x'$ and $y = 3y'$,

$$\begin{aligned}
 x' &= x'^2 + 3x'^3 + 6x'^4 + 6x'^5 + 12x'^3y' + 30x'^4y' + 18x'^2y'^2 \\
 &\quad + 78x'^3y'^2 + 96x'^2y'^3 + 132x'y'^4 + 132y'^5, \\
 y' &= x'^4 + 2x'^5 + 4x'^3y' + 13x'^4y' + 32x'^3y'^2 + 88x'^2y'^3 \\
 &\quad + 22y'^4 + 220x'y'^4 + 186y'^5,
 \end{aligned}$$

which coincides with the fixed point equation for $\vec{\Phi}$ in [Ref. 8, (33)] with $x \mapsto x'$ and $y \mapsto y'$. A motivation of the conditions in Theorem 2 was an attempt to generalize the known examples (5).

The class of W allowed by the conditions in Theorem 2 is a subset of that in Theorem 1. This may be easily seen from the following equivalent conditions to those in Theorem 2.

Proposition 4. *The conditions on W in Theorem 2 is equivalent to the following:*

- (i) *The conditions in Theorem 1 hold.*
- (ii) *Each term has total degree no more than 6.*
- (iii) *Terms containing positive powers of y has total degree 5 or 6.*
- (iv) *xy^4 and x^2y^3 are absent.*

That these conditions imply those in Theorem 2 is easily seen, if one notices that the extra conditions in Proposition 4 implies (4) modulo the coefficient of x^4y , which is fixed by the condition $R(x, z)/Y(x, x^2z) = O(x)$. The converse is proved in a similar way.

In Refs. 5 and 8, the results in Theorem 1 and Theorem 2 are proved for $W = W_3$ and $W = W_4$ in (5), respectively, but the proofs there explicitly uses the explicit values of coefficients in W_3 and W_4 . These values of the coefficients are essentially the numbers of certain figures (self-avoiding paths) on the 3 and 4 dimensional gaskets, respectively, and the non-Markovian nature of self-avoiding paths makes it hard to count these numbers, not to mention to find general formula for all d dimensional gaskets.

There is a long history of interests and studies in self-avoiding paths on the Sierpiński gasket and its higher dimensional analogs (d dimensional gaskets). See Refs. 1, 9, 10 and references therein for earlier studies, and⁽²⁻⁸⁾ for detailed rigorous studies of the subject. The RG approach is crucial in these studies, because the RG maps of self-avoiding paths on d dimensional gaskets are finite dimensional, hence well-definedness is obvious, while non-Markovian nature of the self-avoiding walks and lack of translational invariance of d dimensional gaskets restrict effectiveness of other standard and powerful tools for analysis. Through RG analysis, we know, for example, that existence of relevant fixed point (the one we are looking for in this paper) leads to continuum limit construction of continuous time self-similar process with non-trivial fine structures (e.g., Hausdorff dimension greater than 1),^(2,3,5,8) and existence of convergent trajectory of RG to the relevant fixed point implies asymptotic properties such as laws of iterated logarithms.⁽⁴⁻⁸⁾

A problem in the current RG approach is, as mentioned above, that one has to explicitly classify and count the number of self-avoiding paths to find a RG map. It is therefore important for the RG approach that the results in the above theorems could be derived from ‘basic properties’ for W which can be derived by simple arguments. It is not very difficult to derive the conditions in Proposition 4 (including those in Theorem 1) from basic graphical considerations in the case

of restricted self-avoiding paths on 3 and 4 dimensional gaskets, so Theorem 2 provides rather satisfactory alternative proofs to the corresponding original proofs in Refs. 5, 8, in that one no more needs to count the number of self-avoiding paths exactly, for a proof of existence and uniqueness of fixed points in Ξ^o . We will, however, avoid further details on self-avoiding paths or gaskets in this paper, in order to emphasize that all the results and the proofs in this paper logically requires no knowledge from these subjects, and are solely based on analysis of the maps.

We also note that the examples (5) do not seem to fit to any existing general theorems on fixed point uniqueness, much less the class in Theorem 2. This may reflect the fact that self-avoiding paths are non-Markovian and mathematically hard to analyze. The present study may then provide a new direction in the study of fixed point theorems.

A plan of this paper is as follows. In Sec. 2 we prove Theorem 1 and in Sec. 3 we prove Theorem 2.

We note that our proof for Theorem 2 in Sec. 3 in fact proves a stronger property than is stated in the Theorem, and does not hold for all W in the class satisfying the conditions in Theorem 1. However, it seems that even for examples where the proof in Sec. 3 breaks down, the statement in Theorem 2 still seems to hold. We therefore close this Introduction with the following conjecture.

Conjecture 5. *Uniqueness of the fixed point (x_f, y_f) of $\Phi = \text{grad } W$ in Ξ^o hold under the condition in Theorem 1.*

2. PROOF OF EXISTENCE OF FIXED POINT

Here we prove Theorem 1.

Let $W = W(x, y)$ be a polynomial with non-negative coefficients satisfying the conditons in Theorem 1, and define functions in 2 variables F and G by

$$G(x, z) = \frac{1}{x}X(x, x^2z), \quad (6)$$

and

$$F(x, z) = z \frac{X^2(x, x^2z)}{Y(x, x^2z)}, \quad (7)$$

where $\text{grad } W = (X, Y)$.

Note that with the change of variables $(x, y) \mapsto (x, z)$ defined by $y = x^2z$, the set $\Xi \setminus \{0\} = \{(x, y) \in \mathbb{R}_+^2 \mid y \leq x^2\} \setminus \{0\}$ is mapped to a strip in the first quadrant \mathbb{R}_+^2

$$\tilde{\Xi} = \{(x, z) \in \mathbb{R}^2 \mid x > 0, 0 \leq z \leq 1\}. \quad (8)$$

Since $W(x, y)$ has a term of a form $x^n y$ by a condition in Theorem 1, $Y(x, x^2 z) > 0$, $(x, z) \in \tilde{\Xi}$, hence $F(x, z)$ is well-defined, positive and analytic on $\tilde{\Xi}$.

Note also that $(x, x^2 z) \in \Xi \setminus \{0\}$ is a fixed point of $\Phi = (X, Y)$ if and only if $(x, z) \in \tilde{\Xi}$ and $F(x, z) = G(x, z) = 1$.

Lemma 6. *$G(x, z)$ is a polynomial in x and z with non-negative coefficients, satisfying*

$$G(x, z) = 3ax(1 + O(x)), \text{ as } x \rightarrow 0, \tag{9}$$

where a is the coefficient of the term x^3 in W . Furthermore, the contour set for $G = 1$ in the strip $\tilde{\Xi}$ is a smooth curve connecting the floor $z = 0$ and the ceiling $z = 1$; Namely, there exists a positive continuously differentiable function $x^*(z) > 0$ for $0 \leq z \leq 1$ such that

$$\{(x, z) \in \tilde{\Xi} \mid G(x, z) = 1\} = \{(x^*(z), z) \mid 0 \leq z \leq 1\}. \tag{10}$$

F satisfies $F(x, z) > 0$ for all $(x, z) \in \tilde{\Xi}^\circ$, and $F(x, 0) = 0$ and $F(x, 1) > 1$ for $x > 0$.

Proof: All the statements about G are obvious from the conditions in Theorem 1, except perhaps the last one. To see that the stated x^* exists, first note that by the conditions in Theorem 1, W is a polynomial with non-negative coefficients with lowest order being x^3 , hence G is a polynomial with non-negative coefficients satisfying $\partial G / \partial x(x, z) \geq 3a > 0$, $\lim_{x \downarrow 0} G(x, z) = 0$, and $\lim_{x \rightarrow \infty} G(x, z) = \infty$ for $0 \leq z \leq 1$. This with an implicit function theorem implies that there uniquely exists a continuously differentiable function $x^* : [0, 1] \rightarrow \mathbb{R}_{>0}$ such that $G(x^*(z), z) = 1$, $0 \leq z \leq 1$, and monotonicity of G in x implies that every point satisfying $G(x, z) = 1$ is on the curve $\{(x^*(z), z)\}$. Statements on F are also easy. □

A proof of Theorem 1 is now obvious, because Lemma 6 implies $F(x^*(0), 0) = 0$ and $F(x^*(1), 1) > 1$, for a smooth curve $\{(x^*(z), z) \mid 0 \leq z \leq 1\} \subset \tilde{\Xi}$ hence there is a $z^* \in (0, 1)$ such that $F(x^*(z^*), z^*) = G(x^*(z^*), z^*) = 1$, which, as noted at the beginning of this section, implies the existence of a fixed point of $\text{grad } W = (X, Y)$ in Ξ° . □

3. PROOF OF UNIQUENESS OF FIXED POINT

Here we prove Theorem 2.

Let $J_{GF}(x, z)$ be the Jacobian matrix of the map $(x, z) \mapsto (G(x, z), F(x, z))$;

$$J_{GF} = \frac{\partial G}{\partial x} \frac{\partial F}{\partial z} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial z}. \quad (11)$$

A core of our proof of Theorem 2 is to prove $J_{GF} \neq 0$ on the contour curve $G = 1$ in $\tilde{\Xi}$. This implies that the map is locally one-to-one, which further implies, with additional properties such as (10) and Lemma 7 below, global one-to-one properties, implying uniqueness of the fixed point.

The proof of Theorem 2 in this section starts with Lemma 7 then follows Lemma 8 where we prove that, with these properties, positivity of Jacobian J_{GF} is sufficient for a proof of Theorem 2. Up to this point, the arguments are ‘soft’ and all the results hold for the class in Theorem 1. The hardest part comes last, a proof that $J_{GF} > 0$ for the class assumed in Theorem 2. That this is hard may be seen if one notices that outside $\tilde{\Xi}$ there may be more than one fixed points (as are the cases for the examples below Theorem 1 and (5)^(5,8)), hence $J_{GF} < 0$ actually occurs for some $(x, z) \in \mathbb{R}_+^2$. We must therefore find a nice quantity which is explicitly positive only in a subset of $\tilde{\Xi}$ and then prove (as we will in Lemma 9) that the quantity is a lower bound of J_{GF} using inequalities in Proposition 3.

Lemma 7. *Under the conditions in Theorem 1, if $x > 0$ is sufficiently small, then*

$$J_{GF}(x, z) > 0, \quad 0 \leq z \leq 1,$$

and furthermore, the set of $(x, z) \in \tilde{\Xi}$ satisfying $F(x, z) = 1$ is a single curve for small x , having $(0, 1)$ as an endpoint. (More precisely, there exists $\delta > 0$ such that $\{(x, z) \in (0, \delta] \times [0, 1] \mid F(x, z) = 1\}$ is a curve whose endpoints are $(0, 1)$ and a point on $x = \delta$.)

Proof: Note that (3) with $F(x, z) = z \left(1 + \frac{R(x, z)}{Y(x, x^2z)}\right)$ implies

$$F(x, z) = z(1 + O(x)), \quad (12)$$

uniformly in $0 \leq z \leq 1$, which, with (9) implies $J_{GF} = 3a + O(x) > 0$ for small x , say $0 < x < \delta$. This in particular implies $\text{grad } F \neq 0$, hence $\{(x, z) \in (0, \delta] \times [0, 1] \mid F(x, z) = 1\}$ is a finite union of non-intersecting smooth curves, each segment of which is either closed or with endpoints at $x = 0$ or $x = \delta$.

Lemma 6 implies $F(x, 1) > 1$ for $x > 0$ and $F(x, 0) = 0$, so that a contour curve for $F = 1$ cannot intersect $z = 0$ nor $z = 1$ for $x > 0$. Also (12) implies $F(+0, z) = z$, which, with $F(x, 1) > 1$ for $x > 0$ implies that a contour curve for $F = 1$ exists and intersects $x = 0$ at $z = 1$. By definition,

$$\begin{aligned} F(x, 1-u) - 1 &= (1-u) \frac{X^2}{Y}(x, x^2(1-u)) - 1 \\ &= \frac{R(x, 1-u) - uX^2(x, x^2(1-u))}{Y(x, x^2(1-u))}, \end{aligned}$$

and

$$\begin{aligned} R(x, 1 - u) - uX^2(x, x^2(1 - u)) \\ &= R(x, 1) - 9a^2x^4(1 + O(x))u + x^7 O_x(u^2) \\ &= O(x^5) - 9a^2x^4(1 + O(x))u. \end{aligned}$$

Hence, For small x and u , the contour $F(x, 1 - u) = 1$ is uniquely given by $u = O(x)$ in $(x, 1 - u) \in (0, \delta] \times [0, 1]$. \square

Lemma 8. *In addition to the conditions in Theorem 1, assume that $J_{GF}(x, z) \neq 0$ on*

$$\tilde{\Xi}' = \{(x, z) \in (0, \infty) \times (0, 1) \mid G(x, z) \leq 1, F(x, z) \leq 1\} \subset \tilde{\Xi}^o. \quad (13)$$

Then the fixed point of $\Phi = \text{grad } W = (X, Y)$ is unique in Ξ^o .

Proof: As noted at the beginning of Sec. 2, $(x, y) \in \Xi^o$ is a fixed point of Φ if and only if $(x, y/x^2) \in \tilde{\Xi}^o$ and $F(x, y/x^2) = G(x, y/x^2) = 1$. Theorem 1 implies that there is a fixed point $(x_f, y_f) \in \Xi^o$ of Φ . Put $z_f = y_f/x_f^2$.

Then $F(x_f, z_f) = G(x_f, z_f) = 1$, hence $(x_f, z_f) \in \tilde{\Xi}'$.

As in the proof of Lemma 7, $J_{GF} \neq 0$ implies $\text{grad } F \neq 0$, which further implies that

$$A = \{(x, z) \in \tilde{\Xi}' \mid F(x, z) = 1\} \subset \tilde{\Xi}'$$

is a finite union of non-intersecting smooth curves, each segment of which is either closed, or with one endpoint $(0, 1)$ and the other on $\{(x, z) \in \tilde{\Xi}' \mid G(x, z) = 1\}$. (See the next paragraph.) (x_f, z_f) is on one of such curves.

Suppose that (x_f, z_f) is on a smooth curve $C : [0, 1] \rightarrow A$ with

$$\frac{dC}{ds}(s) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 0 < s < 1, \quad (14)$$

and $F(C(s)) = 1, 0 \leq s \leq 1$, whose endpoints are both on $G = 1$; $G(C(0)) = G(C(1)) = 1$ and $C(s) \in \tilde{\Xi}', 0 \leq s \leq 1$. A mean-value Theorem then implies that there exists $s_0 \in (0, 1)$ such that

$$0 = \left. \frac{dG(C(s))}{ds} \right|_{s=s_0} = (\text{grad } G)(C(s_0)) \cdot \frac{dC}{ds}(s_0).$$

$F(C(s)) = 1$ also implies

$$0 = \frac{dF(C(s))}{ds} = (\text{grad } F)(C(s)) \cdot \frac{dC}{ds}(s).$$

Therefore

$$\begin{pmatrix} \frac{\partial F}{\partial x}(C(s_0)) & \frac{\partial F}{\partial z}(C(s_0)) \\ \frac{\partial G}{\partial x}(C(s_0)) & \frac{\partial G}{\partial z}(C(s_0)) \end{pmatrix} \begin{pmatrix} C'_x(s_0) \\ C'_z(s_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where we wrote $\frac{dC}{ds} = \begin{pmatrix} C'_x \\ C'_z \end{pmatrix}$. With (14), we find $J_{GF}(C(s_0)) = 0$, which contradicts the assumption $J_{GF} \neq 0$ on $\tilde{\Xi}'$. Therefore, a contour curve in A on which (x_f, z_f) exists, cannot have both endpoints on $G = 1$. Similarly, such a curve cannot be a closed curve in $\tilde{\Xi}'$. Therefore the curve must have one endpoint $(0, 1)$ and the other on $G = 1$, the latter endpoint being the point (x_f, z_f) .

By Lemma 7, a curve of the contour set $F = 1$ that has endpoint $(0, 1)$ is unique. Therefore there is only one curve $C \subset A \subset \tilde{\Xi}'$ on which there is a point satisfying $F(x_f, z_f) = G(x_f, z_f) = 1$, hence, as noted at the beginning of the proof, the fixed point $(x_f, x_f^2 z_f)$ is unique in Ξ^o . \square

A proof of Theorem 2 is now reduced to proving $J_{GF}(x, z) \neq 0$ on (13) under the conditions in Theorem 2. This follows as the direct consequence of the following Lemma 9. In fact, the Lemma states positivity of $J_{GF}(x, z)$ on

$$\tilde{\Xi}'' = \{(x, z) \in (0, \infty) \times (0, 1) \mid F(x, z) \leq 1\} \subset \tilde{\Xi}^o, \tag{15}$$

which is larger than (13).

Lemma 9. *Assume that W satisfies the conditions in Theorem 2. Let e be a function defined by*

$$e(x, z) = (1 - z)x^2 \frac{Y^2}{X^2}(x, x^2 z) \left(J_{GF} - \frac{F(1 - F)}{z(1 - z)} \frac{\partial G}{\partial x} \right)(x, z). \tag{16}$$

Then $e(x, z)$ is a polynomial in $x, z, 1 - z$ with non-negative coefficients. Namely, there exists a polynomial $f(x, z, s)$ in 3 variables with non-negative coefficients such that $e(x, z) = f(x, z, 1 - z)$. Furthermore, $f(x, z, s)$ has a term $a^4 s^2 z x^9$, hence in particular, $e(x, z) > 0$, and consequently, $J_{GF}(x, z) > 0, (x, z) \in \tilde{\Xi}''$.

Proof: The last claim is by explicit calculation of order x^9 terms. See the term $C[9, z, s]$ in Appendix. For this and for the calculations below, we use Mathematica software to assist the simple algebraic manipulation such as expanding and factoring. ($e(x, z)$ has more than 300 terms with positive coefficients and more than 80 terms with negative ones!)

The problem is to use the R_n s in Proposition 3 to eliminate apparent negative signs in $e(x, z)$. It turns out that we have an expression

$$e(x, z) = e_c(x, z, 1 - z) + e_r(x, z, 1 - z) \tag{17}$$

where

$$\begin{aligned}
 e_c(x, z, s) = & 3a R_5 z x^7 + 3a R_6 z x^8 + 8b R_5 z x^8 + 3a R_7 z^2 (1+s)x^9 \\
 & + 8b R_6 z x^9 + 15 f_5 R_5 z x^9 + 3a R_8 z^3 (1+2s)x^{10} \\
 & + 8b R_7 z^2 (1+s)x^{10} + 15 f_5 R_6 z x^{10} + R_5 (a^2 (144z^3 + 36z^3 s) \\
 & + f_6 24z)x^{10} + 8b R_8 z^3 (1+2s)x^{11} + 15 f_5 R_7 z^2 (1+s)x^{11} \\
 & + R_6 (6a^2 z^3 (18 + 6z + 6zs) + f_6 24z)x^{11} + R_5 (g_5 (3z^2 \\
 & + 45z^2 s + 6z^4 + 16z^5) + h_3 (4z^2 + 12z^2 s^2 + 11z^5) \\
 & + ab 24 s^2 z^2 (8+z))x^{11} + R_9 (3z^4 + 9z^4 s)a x^{11} \\
 & + R_{10} a 6 (1+4s)z^5 x^{12} + R_9 b 8(1+3s)z^4 x^{12} \\
 & + R_7 f_6 24(1+s)z^2 x^{12} + R_8 f_5 (1+2s)z^3 15x^{12} \\
 & + R_7 a^2 18z^3 (3+4z+z^2+4z^2 s)x^{12} + R_6 h_3 (6z^4 + 9z^5)x^{12} \\
 & + R_6 g_5 5z^2 (9s+z^2+5z^2 s+4z^4)x^{12} \\
 & + R_5 h_4 4z^3 (1+8s+z^2+4z^3)x^{12} + R_9 f_5 15(1+3s)z^4 x^{13} \\
 & + R_{10} b 16(1+4s)z^5 x^{13} + R_8 f_6 24z^3 (1+2s)x^{13} \\
 & + R_8 a^2 9z^4 (16+4(1+z)s)x^{13} + R_7 g_5 (25z^3 s + 25z^3 (1+z)s \\
 & + 25z^6)x^{13} + R_5 n_3 3z^3 (10zs+7z^3)x^{13} + R_7 h_3 (5z^5 + 10z^6)x^{13} \\
 & + R_5 a f_6 108z^3 s^3 x^{13} + R_6 h_4 (16z^4 + 8z^4 (1+z)s + 8z^7)x^{13} \\
 & + R_{10} f_5 30z^5 (1+4s)x^{14} + R_9 f_6 24(1+3s)z^4 x^{14} \\
 & + R_9 a^2 18/5 (39+46s)z^5 x^{14} + R_6 n_3 (21z^5 + 3z^6)x^{14} \\
 & + R_7 h_4 22z^5 x^{14} + R_8 g_5 (25z^4 + 8z^4 s + 6z^4 (1+z)s)x^{14} \\
 & + R_8 h_3 3(5+3s)z^6 x^{14} + R_5 a_{24} (10z^5 + 6z^6)x^{14} \\
 & + R_{10} f_6 48z^5 (1+4s)x^{15} + R_{10} a^2 288(1+2s)z^6 x^{15} \\
 & + R_9 h_3 18z^5 x^{15} + R_5 a_{15} 12z^6 x^{15} + R_9 g_5 2/5 (61+94s)z^5 x^{15} \\
 & + R_6 a_{24} 24z^5 x^{15} + R_8 h_4 z^5 (4+20z+20zs)x^{15} \\
 & + R_7 n_3 21z^6 x^{15} + R_8 n_3 z^7 (21+9s)x^{16} \\
 & + R_{10} g_5 (50z^6 + 120z^6 s)x^{16} + R_{10} h_3 (30z^7 + 24z^7 s)x^{16} \\
 & + R_9 h_4 (23z^6 + 8z^6 s)x^{16} + R_7 a_{24} 26z^6 x^{16} + R_6 a_{15} 13z^6 x^{16} \\
 & + R_{10} h_4 (48z^7 + 64z^7 s)x^{17} + R_9 n_3 (17z^7 + z^8 + 7z^8 s)x^{17}
 \end{aligned}$$

$$\begin{aligned}
& + R_8 a_{24} 16 z^8 x^{17} + R_7 a_{15} (2 z^8 + 12 z^9) x^{17} \\
& + R_{10} n_3 (42 z^8 + 24 z^8 s) x^{18} + R_9 a_{24} (3 z^8 + 14 z^9) x^{18} \\
& + R_8 a_{15} (10 z^8 + 5 z^9) x^{18} + R_{10} a_{24} 32 z^9 x^{19} \\
& + R_9 a_{15} (z^9 + 8 z^{10}) x^{19} + R_{10} a_{15} (10 z^{10} + 8 z^{11}) x^{20},
\end{aligned}$$

with which the remainder $e(x, z) - e_c(x, z, 1 - z)$ has an expression $e_r(x, z, 1 - z)$ for a polynomial $e_r(x, z, s)$ with non-negative coefficients, where non-negativity of e_r holds by non-negativity of a, b, \dots , without the conditions $R_n \geq 0$. For completeness, we give an explicit form of e_r in the appendix.

By (17), $e(x, z) = f(x, z, 1 - z)$ with $f(x, z, s) = e_c(x, z, s) + e_r(x, z, s)$, which completes a proof of Lemma 9, hence, as noted at the beginning of this section, a proof of Theorem 2 is also complete. \square

We remark that Lemma 9 is a result much stronger than is required to prove Theorem 2. In fact, with Lemma 9, a similar argument as for the contour curves $F = 1$ and $G = 1$ hold for contours $G = c$ for any $c > 0$ and $F = c$ for any $c \leq 1$, hence in particular, we have the following.

Corollary 5. $\tilde{\Xi}''$ defined by (15) is a connected set, whose boundary is $\{x = 0\} \cup \{z = 0\} \cup \{F = 1\}$, and the map $(x, z) \mapsto (G, F)$ is globally one-to-one on $\tilde{\Xi}''$.

We also remark that the formula (16) and the rather lengthy e_c was found to work by trial and error, and it is an open problem to find their intuitive (either mathematical or physical) meaning.

APPENDIX: AN EXPLICIT FORM OF e_r

For completeness, we will give an explicit form of e_r defined in the proof of Lemma 9. (Note that it is not unique. For example, there is more than one way of writing $3 - 2z$ as a polynomial of z and $1 - z$ with positive coefficients; $3 - 2z = 1 + 2(1 - z) = z + 3(1 - z)$.)

$$e_r(x, z, s) = \sum_{n=9}^{30} C[n, z, s] x^n, \text{ where,}$$

$$C[9, z, s] = 12 a (54 a^3 + 10 b f_5 + 9 a f_6) s^2 z,$$

$$C[10, z, s] = s^2 z (320 b^2 f_5 + 144 a b (18 a^2 z + f_6 (3 + 2z))$$

$$+ 3 a (25 f_5^2 (1 + 2z) + 3 (22 a g_5 + 16 a g_5 z + 4 a h_3 z + 5 a_{05} z^2))),$$

$$\begin{aligned}
C[11, z, s] = & s^2 z (15 g_5^2 + 38 g_5^2 z + 110 g_5 h_3 z + 32 h_3^2 z + 16 g_5^2 z^2 \\
& + 43 g_5 h_3 z^2 + 22 h_3^2 z^2 + 144 a^2 h_4 z (1 + z) \\
& + 2160 a^3 f_5 z (1 + 2z) + 384 b^2 (f_6 + 2 f_6 z) \\
& + 180 a f_5 f_6 (4 + 2z + 3z^2) + 40 b (3 a_{05} z^2 + 10 f_5^2 (2 + z))),
\end{aligned}$$

$$\begin{aligned}
C[12, z, s] = & s^2 z (225 a_{05} f_5 z^2 + 432 b^2 h_3 z^2 + 360 a f_5 h_3 z^3 \\
& + f_5^3 (375 + 750 z) + b f_5 f_6 (2160 + 2400 z + 1440 z^2) \\
& + b^2 g_5 (400 z^2 + 320 z^2 s) + h_3 h_4 (90 z^2 + 32 z^3) \\
& + g_5 h_4 (104 z + 66 z^2 + 56 z^3) + a f_6^2 (972 + 216 z + 324 z^2 \\
& + 432 z^3) + a^2 b f_5 (12960 z + 10800 z^2 + 2880 z^3) \\
& + a^3 f_6 (1296 z + 5832 z^2 + 7776 z^3) + a^5 (23328 z^2 + 31104 z^3)),
\end{aligned}$$

$$\begin{aligned}
C[13, z, s] = & z (72 a b n_3 z^5 + a_{05} f_6 360 s^2 z^2 + f_5^2 f_6 300 s^2 (5 + 10 z + 9 z^2) \\
& + a^3 g_5 1080 z^4 (1 + 4 s) + a^3 h_3 216 z^3 (3 + s (12 + 7 z)) \\
& + a f_6 g_5 36 z (5 z^4 + 25 z^3 s + 35 s^2 + s^3 (15 + 8 z)) \\
& + a f_6 h_3 36 z^3 ((3 + 16 z) s + 3) + a^2 a_{24} (36 z^4 + 144 z^4 s) \\
& + h_3 n_3 72 s^2 z^3 + g_5 n_3 3 s z^2 (32 s (1 + z) + 2 + 3 z) \\
& + b f_5 h_3 80 s z^2 (5 + s (10 + 7 z)) \\
& + b f_5 g_5 200 s^2 z (10 + 10 z + 3 z^2) + a^2 a_{05} 900 s^2 z^3 \\
& + a^2 f_5^2 900 s^2 z (10 + 20 z + z^2) + b f_6^2 288 s^2 (5 + 10 z + 3 z^2 + 4 z^3) \\
& + a^2 b f_6 864 s^2 z (20 + 13 z + 21 z^2) + a^4 b 51840 s^2 z^2 (1 + z) \\
& + a f_5 h_4 120 z^2 (1 + 2 z^3 s + 9 s^2) + b^2 h_4 128 s^2 z^2 (5 + 6 z + z s) \\
& + h_4^2 16 s^2 z^2 (1 + z + z^2)),
\end{aligned}$$

$$\begin{aligned}
C[14, z, s] = & z/5 (a f_6 h_4 360 z^2 (3 + 6 s + 13 s^2) + a a_{24} b (720 z^4 + 720 z^4 s) \\
& + b^2 n_3 240 z^3 (3 + s (8 + z)) + a_{24} g_5 90 z^4 s + h_4 n_3 30 z^3 s^2 \\
& + a g_5^2 75 z^2 (5 (1 + s)^2 + s^2 (35 + 12 z)) + a_{05} g_5 975 s^2 z^3 \\
& + b f_6 g_5 240 z (4 z^3 + 6 z^4 + 46 s + (9 + 64 z) s^2) \\
& + a f_5 n_3 4950 s^2 z^3 + b f_5 h_4 400 z^2 (3 z^2 + 2 z^3 + 22 s + 22 z s^2) \\
& + f_5 f_6^2 900 s^2 (11 + 22 z + 33 z^2 + 12 z^3) \\
& + f_5^2 g_5 125 s^2 z (55 + 110 z + 126 z^2)
\end{aligned}$$

$$\begin{aligned}
&+ a^3 h_4 864 z^3 (8 + 24 s + 23 s^2) \\
&+ a^2 b g_5 720 z^2 (21 z^3 + 105 s + (5 + 37 z) s^2) \\
&+ a^2 f_5 f_6 1080 z (z^4 + z^2 5 s + 55 s^2 (2 + 4 z + 3 z^2)) \\
&+ a^4 f_5 6480 z^2 (z^3 + z^2 5 s + 55 (1 + 2 z) s^2) + a_{24} h_3 180 z^5 s \\
&+ a h_3^2 135 z^4 (1 + 4 s + 6 s^2) + f_5^2 h_3 375 s^2 z^2 (11 + 22 z + 3 z^2) \\
&+ a g_5 h_3 90 z^3 (5 z^3 + 30 z^2 s + 8 s^2 (2 + 5 z)) \\
&+ b f_6 h_3 720 z^2 (2 + 2 s (1 + z^2) + s^2 (7 + 3 z^2)) \\
&+ a^2 b h_3 432 z^3 (21 z^2 + 21 5 s + 5 s^2 (1 + 6 z)),
\end{aligned}$$

$$\begin{aligned}
C[15, z, s] = & z/5 (b f_5 n_3 (3000 z^3 + 4200 z^3 (1 + z) s) + n_3^2 45 z^4 s \\
&+ b g_5 h_3 96 z^3 (7 z^2 + 7 5 s + 40 s^2) + a f_6 n_3 540 z^3 (z^2 + z 4 s \\
&+ 12 s^2) + b g_5^2 80 z^2 (14 z^3 + 70 z^2 s + 75 s^2 (1 + 2 z)) \\
&+ f_5 f_6 h_3 1800 z^2 (z^2 + 4 z^2 s + 6 s^2 (1 + 2 z)) \\
&+ f_6^3 4320 s^2 (1 + 2 z + 3 z^2 + 4 z^3) + f_5 f_6 g_5 360 z (13 z^4 \\
&+ 65 z^3 s + 50 s^2 (1 + 2 z + 3 z^2)) + a_{24} h_4 (80 z^5 + 400 z^5 s) \\
&+ a_{15} h_3 (105 z^4 + 15 s z^4 (23 + 5 z)) + a_{05} h_4 100 z^4 s^2 \\
&+ a h_3 h_4 1800 z^4 s^2 + f_5^2 h_4 500 z^2 (6 + 6 s (1 + z) + 5 z^2 s^2) \\
&+ a g_5 h_4 24 z^3 (28 z^2 + 28 5 s z + 25 s^2 (12 + 5 z)) \\
&+ b f_6 h_4 960 z^2 (2 z^4 + 12 s + 3 s^2 z (4 + z)) \\
&+ a^3 n_3 3240 z^4 (z + 4 s + 8 s^2) + a^2 f_5 h_3 10800 z^3 (z^3 \\
&+ 3 (1 + s) s (1 + z)) + a^2 f_6^2 25920 s^2 z (3 + 6 z + 9 z^2 + 2 z^3) \\
&+ a^2 f_5 g_5 720 z^2 (39 z^3 + 5 39 s z^3 + 5 s^2 (30 + 60 z + 59 z^2)) \\
&+ a^2 b h_4 5760 z^3 (2 + 6 s + s^2 (4 + 13 z)) \\
&+ a^4 f_6 155520 s^2 z^2 (3 + 6 z + 4 z^2) + a^6 933120 s^2 z^3 (1 + 2 z)),
\end{aligned}$$

$$\begin{aligned}
C[16, z, s] = & z^2 (3 a_{15} h_4 z^4 + a_{24} b f_5 1040 z^3 s + a a_{24} f_6 936 z^3 s \\
&+ a h_4^2 24 z^3 (3 + 15 s + 8 s^2) + f_5 f_6 h_4 (29 60 z^2 \\
&+ 60 (3 + s) s z (13 + 9 z + 8 z^2)) + f_6^2 g_5 180 (8 (1 + s z) \\
&+ s^2 (5 + 2 z + 7 z^2 + 12 z^3)) + b g_5 h_4 40 z^2 (9 z^2 + 48 s \\
&+ 4 s^2 (1 + 14 z)) + a^2 f_5 h_4 360 z^2 (29 z^3 + s (52 + 52 z + 21 z^2))
\end{aligned}$$

$$\begin{aligned}
& + f_5 g_5^2 125 z (3 + 6 s + 2 s^2 (2 + 7 z + 12 z^2)) \\
& + a^2 f_6 g_5 2160 z (8 + 3 s + s^2 (2 + 7 z + 12 z^2)) \\
& + a^4 g_5 6480 z^2 (8 z + s (1 + 5 z) + 12 s^2 (1 + z)) \\
& + a_{24} n_3 (30 z^4 + 36 z^4 (1 + z) s) + f_5^2 n_3 75 z^2 (6 + 18 s z^2 \\
& + s^2 (7 + 14 z + 3 z^2)) + a g_5 n_3 90 z^3 (1 + s (3 + z) (1 + 3 s)) \\
& + f_5 h_3^2 45 z^3 (3 + 12 s z + 2 s^2 (5 + 4 z)) \\
& + a h_3 n_3 54 z^4 (1 + 5 s + 7 s^2) + b h_3 h_4 24 z^3 (9 z + 48 s \\
& + 4 s^2 (1 + 4 z)) + b f_6 n_3 144 z^2 (6 + 18 s z^2 + s^2 (7 + 14 z + 3 z^2)) \\
& + f_6^2 h_3 108 z (1 + z) (4 z^4 + 13 s (1 + z^2)) \\
& + f_5 g_5 h_3 150 z^2 (3 z^3 + 9 s (1 + z) + 4 s^2 (1 + z)^2) \\
& + a^2 b n_3 864 z^3 (1 + z) (3 z^2 + 13 s) \\
& + a^2 f_6 h_3 1296 z^2 (8 z^4 + s (13 + 13 z + 13 z^2 + 4 z^3)) \\
& + a^4 h_3 3888 z^3 (1 + z) (4 z^2 + 13 s),
\end{aligned}$$

$$\begin{aligned}
C[17, z, s] = & z^3 (2100 f_6 g_5^2 + 12600 a^2 g_5^2 z + 2520 f_6 g_5 h_3 z \\
& + 15120 a^2 g_5 h_3 z^2 + 756 f_6 h_3^2 z^2 + 4536 a^2 h_3^2 z^3 \\
& + a^2 a_{24} b (3456 z^3 + 4608 z^3 s (1 + z)) \\
& + a_{24} b f_6 (576 z^2 + 768 z^2 s (1 + z + z^2)) \\
& + a a_{24} g_5 (360 z^3 + 480 z^3 s (1 + z)) \\
& + a_{24} f_5^2 (300 z^2 + 400 z^2 s (1 + z + z^2)) \\
& + a a_{24} h_3 (216 z^4 + 288 z^4 s) + f_5 h_3 h_4 (1680 z^2 s (1 + z) + 960 z^5) \\
& + b h_4^2 128 z^2 (z^3 + 7 s + 7 s^2 z) + f_6^2 h_4 (2016 s (1 + z + z^2 + z^3) \\
& + 1152 z^5) + f_5 g_5 h_4 (2800 z s (1 + z + z^2) + 1600 z^5) \\
& + a^2 f_6 h_4 (24192 z s (1 + z + z^2) + 13824 z^5) \\
& + a^4 h_4 (72576 z^2 s (1 + z) + 41472 z^5) + a^3 a_{15} 432 s z^4 (7 + 6 z) \\
& + a_{15} b f_5 (80 z^3 s (1 + z) + 480 z^3 s (1 + z + z^2)) \\
& + a a_{15} f_6 (72 z^3 s (1 + z) + 432 z^3 s (1 + z + z^2)) + a_{15} n_3 z^4 s^2 \\
& + b h_3 n_3 24 z^3 (24 z^3 + s (42 + 25 z + 17 z^2)) \\
& + a h_4 n_3 24 z^3 (6 z^4 + s (7 + 3 z) (1 + 5 s + 2 z^2))
\end{aligned}$$

$$\begin{aligned}
&+ b g_5 n_3 40 z^2 (24 z^4 + s (42 + 42 z + 25 z^2 + 17 z^3)) \\
&+ f_5 f_6 n_3 60 z (24 z^5 + s (42 + 42 z + 42 z^2 + 25 z^3 + 17 z^4)) \\
&+ a^2 f_5 n_3 360 z^2 (24 z^4 + s (42 + 42 z + 25 z^2 + 17 z^3)),
\end{aligned}$$

$$\begin{aligned}
C[18, z, s] = & z^4 (625 g_5^3 + 3600 f_6 g_5 h_4 + 1125 g_5^2 h_3 z + 21600 a^2 g_5 h_4 z \\
&+ 2160 f_6 h_3 h_4 z + 1200 f_5 h_4^2 z + 675 g_5 h_3^2 z^2 \\
&+ 12960 a^2 h_3 h_4 z^2 + 135 h_3^3 z^3 + a^2 a_{24} f_5 (4680 z^2 + 1080 z^2 s (1 + z) \\
&+ 5040 z^2 s (1 + z + z^2)) + a^2 a_{15} b (2880 z^3 s + 1440 z^3 s (1 + z)) \\
&+ a_{24} f_5 f_6 (780 z + 180 z s (1 + z + z^2) + 840 z s (1 + z + z^2 + z^3)) \\
&+ a_{24} b g_5 (520 z^2 + 120 z^2 s (1 + z) + 560 z^2 s (1 + z + z^2)) \\
&+ a_{24} b h_3 (312 z^3 + 72 z^3 s + 336 z^3 s (1 + z)) \\
&+ a a_{24} h_4 (312 z^3 + 72 z^3 s + 336 z^3 s (1 + z)) \\
&+ a_{15} b f_6 (480 z^2 s (1 + z) + 240 z^2 s (1 + z + z^2)) \\
&+ a a_{15} g_5 (300 z^3 s + 150 z^3 s (1 + z)) + a_{15} f_5^2 (250 z^2 s (1 + z) \\
&+ 125 z^2 s (1 + z + z^2)) + a a_{15} h_3 90 z^4 s + a_{15} a_{24} (27 z^4 + 2 z^5) \\
&+ a n_3^2 27 z^3 (1 + 2s)(1 + 4s) + f_5 h_3 n_3 90 z^2 (1 + z)(15s + 4z^2) \\
&+ b h_4 n_3 96 z^2 (1 + z)(15s + 4z^2) + f_6^2 n_3 108 (1 + z)(15s(1 + z^2) \\
&+ 4z^4) + f_5 g_5 n_3 150 z (8z^4 + s(15 + 15z + 15z^2 + 4z^3)) \\
&+ a^2 f_6 n_3 1296 z (4 + 11s(1 + z + z^2) + 4z^4) \\
&+ a^4 n_3 3888 z^2 (1 + z)(15s + 4z^2) + a_{05} a_{15} 5 z^4 (2 + z)(2 + 3s)),
\end{aligned}$$

$$\begin{aligned}
C[19, z, s] = & z^5 (1600 g_5^2 h_4 + 1536 f_6 h_4^2 + 2880 f_6 g_5 n_3 + 1920 g_5 h_3 h_4 z \\
&+ 9216 a^2 h_4^2 z + 17280 a^2 g_5 n_3 z + 1728 f_6 h_3 n_3 z \\
&+ 1920 f_5 h_4 n_3 z + 576 h_3^2 h_4 z^2 + 10368 a^2 h_3 n_3 z^2 \\
&+ 576 b n_3^2 z^2 + a^4 a_{24} 20736 (z^2 + z^2 s (1 + z)) \\
&+ a^2 a_{24} f_6 6912 (z + z s (1 + z + z^2)) \\
&+ a_{24} f_5 g_5 800 z (1 + s(1 + z + z^2)) \\
&+ a_{24} f_6^2 576 (1 + s(1 + z + z^2 + z^3)) + a_{24} b h_4 512 z^2 (1 + s(1 + z)) \\
&+ a_{24} f_5 h_3 480 z^2 (1 + s(1 + z)) + a a_{24} n_3 288 z^3 (1 + s)
\end{aligned}$$

$$\begin{aligned}
& + a^2 a_{15} f_5 (2520 z^2 + 360 z^2 s (1 + z) + 2880 z^2 s (1 + z + z^2)) \\
& + a_{15} f_5 f_6 (420 z + 60 z s (1 + z + z^2) + 480 z s (1 + z + z^2 + z^3)) \\
& + a_{15} b g_5 (280 z^2 + 40 z^2 s (1 + z) + 320 z^2 s (1 + z + z^2)) \\
& + a_{15} b h_3 (168 z^3 + 24 z^3 s + 192 z^3 s (1 + z)) \\
& + a a_{15} h_4 (168 z^3 + 24 z^3 s + 192 z^3 s (1 + z)),
\end{aligned}$$

$$\begin{aligned}
C[20, z, s] = & z^6 (2040 a_{24} f_6 g_5 + 1360 g_5 h_4^2 + 1275 g_5^2 n_3 + 2448 f_6 h_4 n_3 \\
& + 12240 a^2 a_{24} g_5 z + 1224 a_{24} f_6 h_3 z + 1360 a_{24} f_5 h_4 z \\
& + 816 h_3 h_4^2 z + 1530 g_5 h_3 n_3 z + 14688 a^2 h_4 n_3 z + 765 f_5 n_3^2 z \\
& + 7344 a^2 a_{24} h_3 z^2 + 816 a_{24} b n_3 z^2 + 459 h_3^2 n_3 z^2 + 204 a a_{24}^2 z^3 \\
& + a^4 a_{15} (10368 z^2 + 6480 z^2 s (1 + z) + 5184 z^2 s (1 + z + z^2)) \\
& + a^2 a_{15} f_6 (3456 z + 2160 z s (1 + z + z^2) + 1728 z s (1 + z + z^2 + z^3)) \\
& + a_{15} f_5 g_5 (400 z + 250 z s (1 + z + z^2) + 200 z s (1 + z + z^2 + z^3)) \\
& + a_{15} f_6^2 (288 + 180 s (1 + z + z^2 + z^3) + 144 s (1 + z + z^2 + z^3 + z^4)) \\
& + a_{15} b h_4 (256 z^2 + 160 z^2 s (1 + z) + 128 z^2 s (1 + z + z^2)) \\
& + a_{15} f_5 h_3 (240 z^2 + 150 z^2 s (1 + z) + 120 z^2 s (1 + z + z^2)) \\
& + a a_{15} n_3 (144 z^3 + 90 z^3 s + 72 z^3 s (1 + z))),
\end{aligned}$$

$$\begin{aligned}
C[21, z, s] = & 1080 a_{15} f_6 g_5 z^7 + 900 a_{24} g_5^2 z^7 + 1728 a_{24} f_6 h_4 z^7 + 384 h_4^3 z^7 \\
& + 2160 g_5 h_4 n_3 z^7 + 972 f_6 n_3^2 z^7 + 6480 a^2 a_{15} g_5 z^8 \\
& + 648 a_{15} f_6 h_3 z^8 + 1080 a_{24} g_5 h_3 z^8 + 10368 a^2 a_{24} h_4 z^8 \\
& + 720 a_{15} f_5 h_4 z^8 + 1080 a_{24} f_5 n_3 z^8 + 1296 h_3 h_4 n_3 z^8 \\
& + 5832 a^2 n_3^2 z^8 + 288 a_{24}^2 b z^9 + 3888 a^2 a_{15} h_3 z^9 + 324 a_{24} h_3^2 z^9 \\
& + 432 a_{15} b n_3 z^9 + 216 a a_{15} a_{24} z^{10},
\end{aligned}$$

$$\begin{aligned}
C[22, z, s] = & 475 a_{15} g_5^2 z^8 + 912 a_{15} f_6 h_4 z^8 + 1520 a_{24} g_5 h_4 z^8 \\
& + 1368 a_{24} f_6 n_3 z^8 + 912 h_4^2 n_3 z^8 + 855 g_5 n_3^2 z^8 + 380 a_{24}^2 f_5 z^9 \\
& + 570 a_{15} g_5 h_3 z^9 + 5472 a^2 a_{15} h_4 z^9 + 912 a_{24} h_3 h_4 z^9 \\
& + 8208 a^2 a_{24} n_3 z^9 + 570 a_{15} f_5 n_3 z^9 + 513 h_3 n_3^2 z^9 \\
& + 304 a_{15} a_{24} b z^{10} + 171 a_{15} h_3^2 z^{10} + 57 a a_{15}^2 z^{11},
\end{aligned}$$

$$C[23, z, s] = 480 a_{24}^2 f_6 z^9 + 800 a_{15} g_5 h_4 z^9 + 640 a_{24} h_4^2 z^9 + 720 a_{15} f_6 n_3 z^9 \\ + 1200 a_{24} g_5 n_3 z^9 + 720 h_4 n_3^2 z^9 + 2880 a^2 a_{24}^2 z^{10} + 400 a_{15} a_{24} f_5 z^{10} \\ + 480 a_{15} h_3 h_4 z^{10} + 4320 a^2 a_{15} n_3 z^{10} + 720 a_{24} h_3 n_3 z^{10} + 80 a_{15}^2 b z^{11},$$

$$C[24, z, s] = 504 a_{15} a_{24} f_6 z^{10} + 420 a_{24}^2 g_5 z^{10} + 336 a_{15} h_4^2 z^{10} + 630 a_{15} g_5 n_3 z^{10} \\ + 1008 a_{24} h_4 n_3 z^{10} + 189 n_3^3 z^{10} + 3024 a^2 a_{15} a_{24} z^{11} \\ + 105 a_{15}^2 f_5 z^{11} + 252 a_{24}^2 h_3 z^{11} + 378 a_{15} h_3 n_3 z^{11},$$

$$C[25, z, s] = 132 a_{15}^2 f_6 z^{11} + 440 a_{15} a_{24} g_5 z^{11} + 352 a_{24}^2 h_4 z^{11} + 528 a_{15} h_4 n_3 z^{11} \\ + 396 a_{24} n_3^2 z^{11} + 792 a^2 a_{15}^2 z^{12} + 264 a_{15} a_{24} h_3 z^{12},$$

$$C[26, z, s] = 115 a_{15}^2 g_5 z^{12} + 368 a_{15} a_{24} h_4 z^{12} + 276 a_{24}^2 n_3 z^{12} \\ + 207 a_{15} n_3^2 z^{12} + 69 a_{15}^2 h_3 z^{13},$$

$$C[27, z, s] = 64 a_{24}^3 z^{13} + 96 a_{15}^2 h_4 z^{13} + 288 a_{15} a_{24} n_3 z^{13},$$

$$C[28, z, s] = 100 a_{15} a_{24}^2 z^{14} + 75 a_{15}^2 n_3 z^{14},$$

$$C[29, z, s] = 52 a_{15}^2 a_{24} z^{15},$$

$$C[30, z, s] = 9 a_{15}^3 z^{16}.$$

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